

# Ternary Frequency-Coded CW Radar Waveform Achieving Almost Perfect Periodic Autocorrelation

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**Abstract**—To broaden the set of available periodic continuous wave (CW) waveforms, a new candidate employing ternary symmetric frequency coding with values  $(-1, 0, +1)$  is proposed. Its periodic autocorrelation function (PACF) closely resembles that of the well-known almost perfect sequence (APS), a binary phase-coded waveform with values  $(-1, +1)$ . Both waveform families exhibit real-valued PACFs with zero sidelobes (SLs), except for a single, negative SL at the midpoint of the period. The binary phase-coded APS family is well-established, with sequence lengths  $N$  generally being multiples of 4. A particularly convenient subfamily, defined by  $N = 2(p + 1)$ , where  $p$  is any odd prime power, can be readily constructed. A transformation method is presented for converting a given APS phase-coded sequence into its frequency-coded counterpart. While a key limitation of frequency coding is that the symmetric spacing of frequency components around 0 is rigidly tied to the code element duration, a significant advantage is that both the transmitted and reference waveforms are unimodular.

**Index Terms**—Almost perfect sequences (APSs), continuous wave (CW) waveforms, frequency coding, periodic autocorrelation, phase coding.

## I. INTRODUCTION

A well-known and significant binary phase-coded continuous wave (CW) radar waveform is the almost perfect sequence (APS) waveform [1], [2], [3], [4]. The importance of APS stems from several key properties.

1) It offers greater diversity than conventional linear frequency modulation (CW-LFM).

2) It employs only two antipodal phase values ( $0^\circ$  and  $180^\circ$ ), making it relatively simple to implement.

3) Its periodic autocorrelation function (PACF) exhibits zero sidelobes (SLs) throughout the period, except for a single SL located at the midpoint.

In contrast, other two-valued phase sequences commonly used for CW radar applications—such as Legendre sequences or m-sequences—require nonantipodal phase values ( $0^\circ, \beta \neq 180^\circ$ ) to achieve perfect PACF, where all SLs are 0 [5]. A recent paper [6], on experimental bistatic CW radar, reported no apparent difficulty in using two nonantipodal phase values. Nevertheless, to avoid the drawbacks associated with nonantipodal phase values, Levanon and Cohen [7] proposed converting a binary phase-coded sequence into a frequency-coded one, resulting in a unimodular CW binary frequency shift keying (BFSK-CW) transmitted waveform.

A limitation of the approach in [7] is that achieving zero delay SLs requires the receiver to employ a mismatched filter that incorporates amplitude keying, thereby introducing a 3-dB mismatch loss.

All phase or frequency-coded CW sequences suffer a major drawback compared with CW-LFM. Processing coded sequences requires capturing the full bandwidth of the complex envelope. In contrast, CW-LFM processing relies only on the much narrower, delay-dependent beat frequency generated between the transmitted signal and the target reflection.

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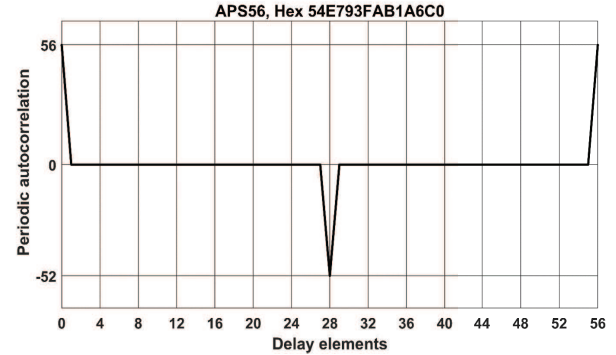


Fig. 1. Periodic autocorrelation of a binary phase-coded APS sequence of length  $N = 56$ , whose hexadecimal representation is 54E793FAB1A6C0.

This correspondence introduces one more periodic CW waveform. The proposed waveform modifies the APS concept by replacing its binary phase-coded modulation with ternary frequency-coded modulation. That addition increases the diversity of frequency-coded CW waveforms, Ashe et al. [8] proposed a phase-to-frequency coding transformation, known as derivative phase (DP), which altered the aperiodic autocorrelation of the compressed pulse. In contrast, our transformation preserves the APS periodic autocorrelation, maintaining a zero imaginary component in the PACF, despite the inherently complex nature of frequency-coded waveforms.

The following section reviews the phase-coded APS waveform, its properties, and known generation methods. Section III introduces the transformation to frequency-coded APS, detailing their properties and comparing both waveform types.

## II. PHASE-CODED APS

Constructing phase-coded APS sequences requires a background in number theory. An early finding established that the number of sequence elements,  $N$ , must be a multiple of 4. An important subgroup of APS requires also that

$$N = 2(p + 1), \quad p = \text{odd prime power.} \quad (1)$$

Recently, constructing an APS belonging to this prime-dependent subgroup has become easier with the introduction of the MATLAB function *apaseq*, available from version 2024a. Currently, *apaseq* generates a single sequence for a given length  $N$ . However, with minor modifications, it can be extended to produce multiple sequences—matching the number of primitive polynomials available for that length.

In an example of APS phase-coded sequence, found in [3], the hexadecimal representation of the  $N = 56$  binary elements is 54E793FAB1A6C0. Its periodic autocorrelation is given in Fig. 1. Because the signal takes real values  $(-1, +1)$ , its autocorrelation is also real. The plot shows a main periodic peak of amplitude  $N$ , followed by zero SLs, except for one, a mid-peak with amplitude  $= -N + 4$ .

For a given length  $N$ , where multiple APS sequences are available, the differences between them are fundamental—extending beyond cyclic shift, horizontal flips or negations. A fundamental difference

**Algorithm 1** Two Long Frequency-Coded Sequences

Table 1. Two halves of the hexadecimal sequence representing the odd frequencies.  $N = 1000$

```
EE05D1E0A4AFB2C29021A4C7E808635AE0537754CF
505AC705DA2E24C019845A97799FF36E2E917C7297
D4335444D7E294E7BFA07369EFDAF2C82B6BE1D17
E205D1E0A4AFB2C29021A4C7E808635AE0537754CF
505AC705DA2E24C019845A97799FF36E2E917C7297
D4335444D7E294E7BFA07369EFDAF2C82B6BE1D17
```

Table 2. Two halves of the hexadecimal sequence representing the odd frequencies.  $N = 1200$

```
2ECEB4684C6C96C3C1FE93F3F9D03B6FAFE98DCA50
3447EF7A0D14EEB4C5955457F71CFBAEC557228831
C4057559734A235D3E8420774FD6B139A028248FD
180C0DA01F0F25B2737A74A3
22CEB4684C6C96C3C1FE93F3F9D03B6FAFE98DCA50
3447EF7A0D14EEB4C5955457F71CFBAEC557228831
C4057559734A235D3E8420774FD6B139A028248FD
180C0DA01F0F25B2737A74A3
```

is recognized when the periodic cross correlation (PCC) between two sequences of the same length resembles PCC between two random sequences.

**III. TRANSFORMATION TO FREQUENCY-CODED APS**

The suggested transformation converts a binary phase-coded APS sequence of length  $N$  into a symmetrical ternary frequency-coded sequence of length  $M = 2N$ . The resulting sequence preserves the almost perfect periodic autocorrelation property, with a main peak of height  $M$ . The three possible values of the normalized frequency  $4f_m t_b$ , assigned to the  $M$  elements, are  $\{-1, 0, +1\}$ , where  $t_b$  denotes the duration of each frequency element.

An additional key property of the normalized frequency-coded sequence is that all even-indexed elements are 0. As a result, all elements at odd-indices  $m = 1, 3, 5, \dots, M-3, M-1$  in the normalized frequency sequence  $4f_m t_b$  take on binary values  $\{-1, +1\}$ .

Since the amplitude is not modulated, the complex envelope over one period is given by

$$u(n) = \exp\left(j \frac{\pi}{2} \sum_{k=1}^n 4f_k t_b\right), \quad 1 \leq n \leq M. \quad (2)$$

It is important to note that the receiver is matched, meaning that the reference waveform used in the receiver is identical to the transmitted waveform.

The binary nature of the odd-indexed frequency elements allows the frequency-coded sequence to be compactly represented as a hexadecimal sequence of  $N/4$  elements. For example, the 56-element phase-coded sequence represented by the 14-element hexadecimal sequence **54E793FAB1A6C0** is transformed into a ternary frequency-coded sequence, where all even elements are 0 and the odd elements are binary  $\{-1, +1\}$ , described by the 14-element hexadecimal sequence **7E945A07E975A0**.

Algorithm 2 presents a MATLAB script that implements an algorithm for transforming the phase-coded sequence into its frequency-coded equivalent, using the hexadecimal sequence referenced above as an example. It also generates a plot of the three components of the PACF of the frequency-coded sequence.

The script requires a cyclic shift value (cs), which is applied to the binary vector  $x$ . A suitable cyclic shift is essential to ensure smooth phase transition between consecutive periods. In the given

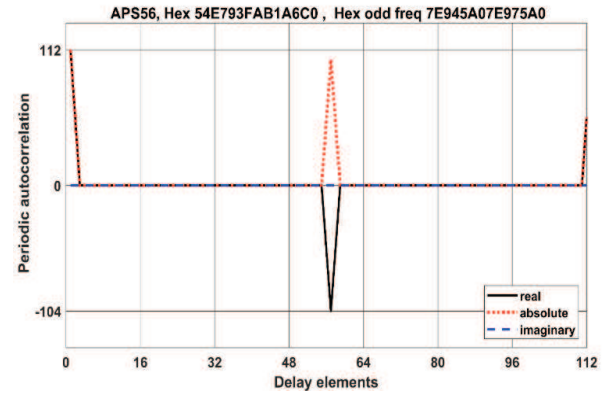


Fig. 2. Periodic autocorrelation of a ternary frequency-coded APS sequence of length  $M = 2N = 112$ , whose hexadecimal odd frequencies representation is **7E945A07E975A0** and  $CS = 0$ .

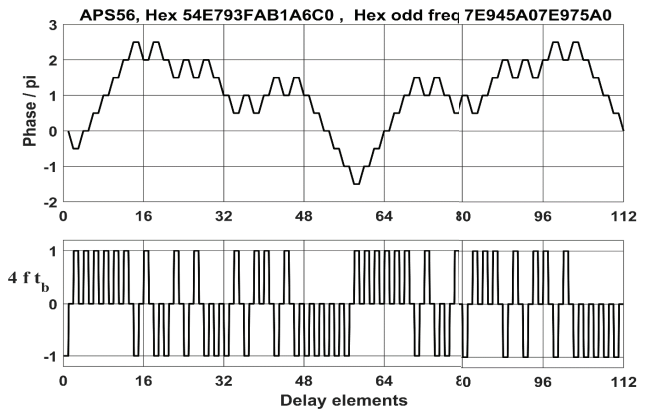


Fig. 3. Frequency coding of a ternary frequency-coded APS sequence whose hexadecimal odd frequencies representation is **7E945A07E975A0** (bottom). Associated phase evolution (top).

example, valid cs values range from 0 to 5. In contrast, values like 6, 7, 9, 10, or 13 disrupt phase continuity, causing imperfect periodic autocorrelation.

The MATLAB script can be downloaded by copying the URL provided in [9] into your browser's address bar and pressing Enter. Alternatively, to extract the script from a PDF edition of this article, use Adobe's conversion tool to convert Algorithm 2 to PowerPoint, then copy the text from the resulting PPTX file and paste it into MATLAB's editor.

Fig. 2 displays the three components—real, imaginary, and absolute—of the periodic autocorrelation of the frequency-coded sequence. All three are necessary because, unlike binary phase coding, frequency coding yields a complex signal envelope, with an autocorrelation function that is typically complex and not restricted to real values. Fig. 3 shows the actual frequency-coded sequence along with the corresponding phase evolution. In the top subplot of Fig. 3 note that one period (112 elements) of the phase evolution starts and ends at the same phase value. The circular shift (cs) is applied specifically to ensure this property.

The hexadecimal representation of the odd-indexed frequencies has a length of  $N/4$ . A noteworthy property, often but not always observed, is that the two halves of this sequence differ by only one hexadecimal element. For example, in our 56-element case the two halves are: **7E945A0** and **7E975A0**. The same property is observed for frequency-coded sequences of length  $N = 1000$  and  $N = 1200$ , as illustrated in Tables 1 and 2 in Algorithm 1.

The bottom subplot of Fig. 3 highlights the required relationship between the three frequencies of the frequency-coded APS complex

**Algorithm 2** MATLAB Script Converting Phase-Coded APS Into a Frequency-Coded Version

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```

clear; nn=56; n_samples=2; nns=nn*n_samples;
hexStr='54E793FAB1A6C0';
binaryVec=hexToBinaryVector(hexStr,nn); x=2*binaryVec- 1; cs=1;
x=circshift(x,cs); dd=0.5; ns=1:n_samples; d_phase=dd/(n_samples-1);
if x(1)>0
    u_phase_q(1,:)=(ns-1)*d_phase;
else
    u_phase_q(1,:)=- (ns-1)*d_phase;
end
for q=2:nn
    if x(q)==x(q-1)
        u_phase_q(q,:)=u_phase_q(q-1,end)-(ns-1)*d_phase;
    else
        u_phase_q(q,:)=u_phase_q(q-1,end)+(ns-1)*d_phase;
    end
end
u_phase_f=reshape(u_phase_q',1,nn*n_samples);
uu=exp(1i*u_phase_f*pi);
f_basic_dp_1=diff(u_phase_f);f_basic_dp=0.5*[f_basic_dp_1 0]; f_basic_list=f_basic_dp;
f_basic_array= (reshape(f_basic_list/0.25, [2,nn]))'; f_basic_line=f_basic_array(:,1)';
binary_f=0.5*(f_basic_line+1); binaryString = num2str(binary_f, '%d'); numBits = 4;
paddedBinaryString=pad(binaryString,ceil(length(binaryString)/numBits)*numBits,'left','0');
groupedBinaryString = reshape(paddedBinaryString, numBits, []).';
hexSequence = cellstr(dec2hex(bin2dec(groupedBinaryString)));
hexResult = strjoin(hexSequence, '');
disp(['Hex of the odd (non-zero) frequencies: ', hexResult]);
out1=real(ifft(fft(uu). *conj(fft(uu)))); out2=abs(ifft(fft(uu). *conj(fft(uu))));
out3=imag(ifft(fft(uu). *conj(fft(uu))));
figure(1), clf, hold off;
p1=plot(out1, 'k', 'linewidth',2); hold on;
p2=plot(out2, 'r:', 'linewidth',2.5); p3=plot(out3, 'b--', 'linewidth',2);
legend([p1 p2 p3], 'real', 'absolute', 'imaginary', 'location', 'southeast'); grid on;
set(gca, 'FontSize',13, 'FontWeight', 'bold', 'GridAlpha',1);
xlim([0, nns]); lim=nns*1.2; lim2=nns; ylim([-lim lim]);
set(gca, 'ytick', [-lim2+8 0 lim2]);
xlabel('Delay elements'); ylabel('Periodic autocorrelation');

```

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envelope and the frequency bit duration  $t_{bf}$ . Note that to ensure equal overall period duration for a corresponding pair of phase-coded and frequency-coded sequences, the duration of each phase element,  $t_{bp}$ , should be twice that of each frequency element,  $t_{bf}$ . In both phase-coded and frequency-coded APS versions, the relevant bit duration influences the delay resolution. However, in frequency-coding  $t_{bf}$  also influences the waveform's frequency design. If the three frequency values  $(1/4 t_{bf})\{-1, 0, +1\}$  are not strictly satisfied, the almost perfect autocorrelation property is compromised.

If the delay resolution  $\Delta t$  is defined as the delay between the autocorrelation peak and its first null, then for the frequency-coded signal  $\Delta t = 2 t_{bf}$ , while for the phase-coded signal  $\Delta t = t_{bp}$ . This difference should be considered when comparing spectra in Figs. 4 and 5. Specifically, the horizontal axis label in Fig. 4 could be replaced with  $f\Delta t/2$ , while in Fig. 5 it could be replaced with  $f\Delta t$ .

In other words, if the horizontal axis label in Fig. 4 were changed to  $f\Delta t$ , the  $x$ -ticks values would double, resulting in identical axis labels and tick values in both figures. With this background, we can conclude that the main spectral lobe of the frequency-coded APS is slightly narrower than that of the phase-coded APS, while the spectral SLs are significantly lower in Fig. 4. A similar observation was reported in [7].

#### IV. DOPPLER RESILIENCE

Doppler resilience of our frequency-coded waveform depends strongly on the period duration and on the number  $P$  of periods coherently integrated. In the presence of target-induced Doppler shift, all three frequency levels are shifted almost equally up or down in frequency. When Doppler is the sole source of disturbance, the periodic ambiguity function (see Fig. 6) illustrates the resulted degradation—primarily observed as increased SL levels with rising Doppler. Fig. 6 corresponds to a TFSK waveform with 176 elements per period and  $P = 8$  coherently processed periods. The delay axis spans up to  $\pm 80$  elements and the normalized Doppler axis extends to  $\nu t_b = \pm 1/14080$ .

Fig. 6 shows that the delay SLs, ideally zero at no-Doppler, rise above  $-60$  dB when  $|\nu M t_b| > 0.03$ ,  $M = 1408$ .

To match a specific Doppler shifted reflection of the signal, the reference sequence in the receiver should incorporate a linear phase-ramp corresponding to the desired Doppler shift. This adjustment shifts the zero delay-SLs region to center around the selected Doppler frequency.

Fig. 7 was introduced to enable comparison with the delay-Doppler response of the corresponding 88-element binary

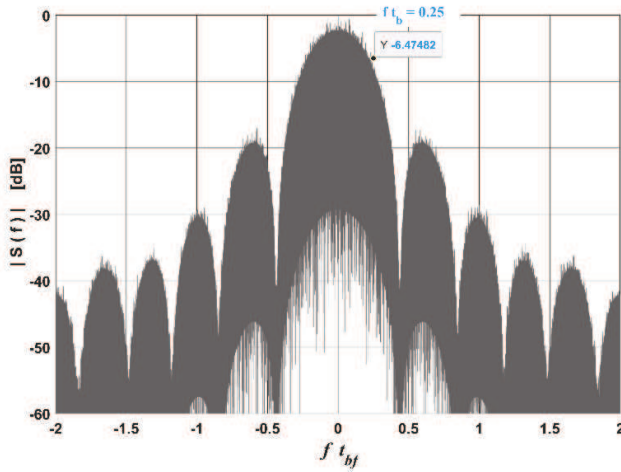


Fig. 4. Spectrum of ternary frequency-coded APS sequence of length  $M = 2N = 2000$ .

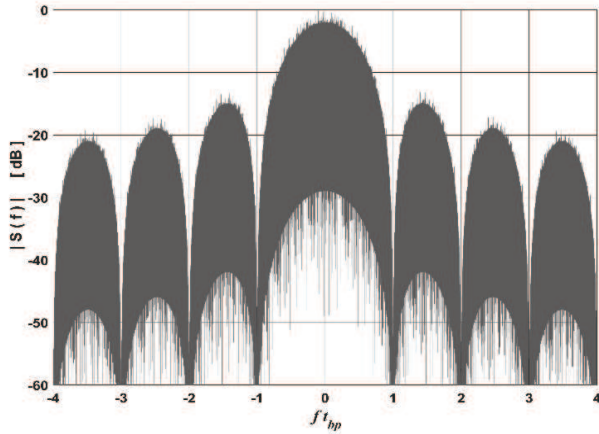


Fig. 5. Spectrum of binary phase-coded APS sequence of length  $N = 1000$ .

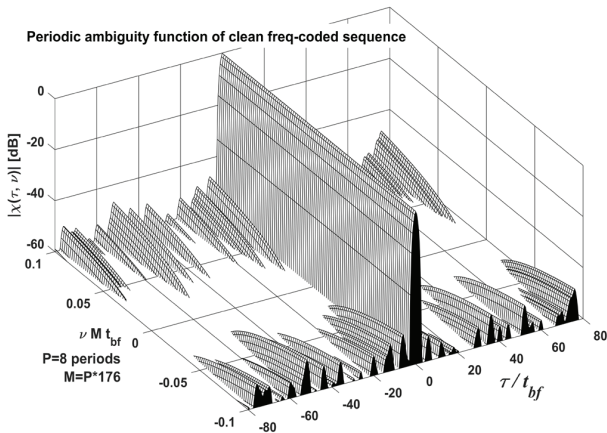


Fig. 6. Delay-Doppler response of  $P = 8$  periods of a 176-element ternary frequency-coded sequence.

phase-coded sequence. The main adverse effect in Fig. 7 is the widening of the main lobe as the magnitude of the Doppler shift increases. The data tip in Fig. 7 highlights the contour line below which this widening becomes evident.

Another valuable comparison between phase-coded and frequency-coded APS waveforms would concern detection performances. However, such a comparison is not practical at this stage, as it depends on numerous parameters related to the waveform, receiver design, target characteristics, and environmental conditions.

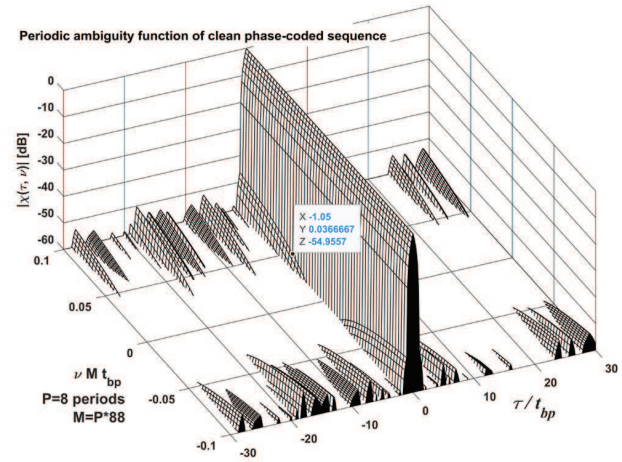


Fig. 7. Delay-Doppler response of  $P = 8$  periods of an 88-element binary phase-coded sequence.

## V. CONCLUSION

A periodic CW unimodular waveform was introduced that utilizes ternary, symmetrical frequency coding. The PACF of the proposed waveform resembles that of the APS, which is a binary (1, +1) phase-coded waveform. Both their PACFs are real and exhibit zero SLs except for one large negative SL, at the period's midpoint.

A transformation is presented in the form of a MATLAB script that converts APS phase-coded sequences into their frequency-coded counterpart. APS sequences are well known and several families can be generated with relative ease.

Numerical comparisons are provided to illustrate the spectral characteristics of both waveform types. The Doppler tolerance of a ternary frequency-coded sequence is also demonstrated. The ability to construct long frequency-coded sequences makes low-power, periodically coded CW waveforms a promising candidate for long-range radar applications.

## REFERENCES

- [1] A. Lempel, M. Cohn, and W. Eastman, "A class of balanced binary sequences with optimal autocorrelation properties," *IEEE Trans. Inf. Theory*, vol. IT-23, no. 1, pp. 38–42, Jan. 1977, doi: [10.1109/TIT.1977.1055672](https://doi.org/10.1109/TIT.1977.1055672).
- [2] H. D. Luke and H. D. Schotten, "Odd-perfect, almost binary correlation sequences," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 31, no. 1, pp. 495–498, Jan. 1995, doi: [10.1109/7.366335](https://doi.org/10.1109/7.366335).
- [3] J. Wolfmann, "Almost perfect autocorrelation sequences," *IEEE Trans. Inf. Theory*, vol. 38, no. 4, pp. 1412–1418, Jul. 1992, doi: [10.1109/18.144729](https://doi.org/10.1109/18.144729).
- [4] W. Van Thillo, P. Gioffré, V. Giannini, D. Guermendi, S. Brebels, and A. Bourdoux, "Almost perfect auto-correlation sequences for binary phase-modulated continuous wave radar," in *Proc. Eur. Microw. Conf.*, Nuremberg, Germany, Oct. 2013, pp. 1803–1806, doi: [10.23919/EuMC.2013.6687029](https://doi.org/10.23919/EuMC.2013.6687029).
- [5] S. W. Golomb, "Two-valued sequences with perfect periodic autocorrelation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 2, pp. 383–386, Apr. 1992, doi: [10.1109/7.144563](https://doi.org/10.1109/7.144563).
- [6] O. Jonsson, R. Ragnarsson, T. Sjögren, T. Thor, and A. Tryblom, "An L-band bistatic radar: Experimental system and measurement campaign," in *Proc. Int. Radar Conf. (RADAR)*, Oct. 2024, pp. 1–6, doi: [10.1109/radar58436.2024.10993560](https://doi.org/10.1109/radar58436.2024.10993560).
- [7] N. Levanon and I. I. Cohen, "Binary frequency shift keying for continuous waveform radar," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 53, no. 5, pp. 2462–2468, Oct. 2017, doi: [10.1109/TAES.2017.2700919](https://doi.org/10.1109/TAES.2017.2700919).
- [8] J. M. Ashe, R. L. Nevin, D. J. Murrow, H. Urkowitz, N. J. Bucci, and J. D. Nesper, "Range sidelobe suppression of expanded/compressed pulses with droop," in *Proc. IEEE Nat. Radar Conf.*, Atlanta, GA, USA, Mar. 1994, pp. 116–122, doi: [10.1109/NRC.1994.328109](https://doi.org/10.1109/NRC.1994.328109).
- [9] *MATLAB Script*. Accessed: Sep. 9, 2025. [Online]. Available: <https://www.dropbox.com/scl/fi/11e06m1aofml6woi6u9rz/phaseToFrequency56.m?rlkey=zabth45iqrc3djj4ltqxc8l1&st=1dm6ordu&dl=1>